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Reduction numbers of equimultiple ideals

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1. Introduction

Let $(A, \mathfrak{m})$ be a local ring and I and an ideal of A. In this paper, all local rings are assumed to have infinite residue fields. An ideal $J \subseteq I$ is called a *reduction* of I if $I^{n+1} = I^nJ$ for some nonnegative integer n. A reduction J is called a *minimal reduction* if it does not properly contain a reduction of I. These notions were introduced by Northcott and Rees [16]. They proved that minimal reductions of I do always exist, and every minimal reduction of I is minimally generated by $l(I) := \dim \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ which is called the *analytic spread* of I. If J is a minimal reduction of I, we define the reduction number of I with respect to J, denoted by $r_J(I)$, to be the last nonnegative integer n such that $I^{n+1} = I^nJ$. The *reduction number* of I is defined by $r(I) = \min\{r_J(I);\ J \text{ is a minimal reduction of } I\}$.

Reduction numbers have been proven to be very useful in studying the Cohen–Macaulay (abbr. C-M) and Gorenstein property of the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and the Rees algebra $A[It] = \bigoplus_{n \geq 0} I^nt^n$ of I. It was initiated by Sally, Goto and Shimoda, and is intensively studied now by various authors (see, e.g., [1] for references). Therefore it is of great interest to study properties of reduction numbers, see [4, 10, 12–14, 19, 23].

The main aim of this paper is to give upper bounds for the reduction numbers of equimultiple ideals. Recall that $\operatorname{ht} I \leq l(I) \leq \dim A$ and I is called an *equimultiple ideal* if $\operatorname{ht} I = l(I)$ [6, 9]. For instance, all $\mathfrak{m}$-primary ideals are equimultiple. Sally [17] gave a bound on $r(\mathfrak{m})$ when A is a C-M ring. Using results of Eakin and Sathaye [4] and of Shalev [20] we can give in Section 3 a bound on the reduction number of an $\mathfrak{m}$-primary ideal in any local ring in terms of a suitable invariant of I. If $d > 1$, easy

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examples show that there is no bound on $r(I)$ depending only on the base ring A even if A is a regular ring. Our bound will involve the order of the ideal I in A (Proposition 3.1). In Section 4 we consider arbitrary equimultiple ideals. We need additional assumptions in order to reduce to the case of $\mathfrak{m}$-primary ideals. This idea was successfully applied in studying the C-M property of $G(I)$ and $A[It]$, see [6, 9, 24]. The condition we need is related to the (projectively) normal Cohen–Macaulayness of A along I (see [9]). In particular, our results extend the corresponding results of Schenzel [19] and Trung [23]. In Section 5 we study the vanishing of certain graded pieces of local cohomology modules of $G(I)$ with support in $G(I)_+$. This is of interest because the reduction number of I is bounded above by the Castelnuovo–Mumford regularity of $G(I)$ (see [23], or Lemma 2.1). We are able to generalize a recent result of Marley [14] to an arbitrary ideal (not necessarily equimultiple) in a local ring (Theorem 5.2). Then we apply it to give a characterization of the C-M property of the Rees algebra $A[It]$ in terms of $G(I)$ and the reduction number $r(I)$. This is a converse of a result of Herrmann et al. [9, Proposition 45.4]. It turns out that if I is an equimultiple ideal and $A[It]$ is a C-M ring, then the reduction number $r(I)$ of I does not depend on the choice of minimal reduction and is bounded above by ht $I-1$ (if A is a C-M ring this is known by results of Goto–Shimoda and of Grothe–Herrmann–Orbanz [6, Theorem 4.8]).

2. Preliminaries

Throughout this paper, if not otherwise stated, let $(A, \mathfrak{m})$ be a d-dimensional local ring and I an ideal of A with the analytic spread $l(I) = s$. Note that dim $G(I) = d$ and $s = \dim G(I)/\mathfrak{m}G(I)$. For short we also use the notation $G = G(I)$ and $R = A[It]$. Given an element $x \in A$, we denote its initial form in $G(I)$ by x^*. A system of elements $x_1, \dots, x_\delta (\delta = \dim A/I)$ is said to be a system of parameters (abbr. s.o.p.) of A modulo I if their images in A/I form a s.o.p. of A/I. **Z** (resp. **N**) denotes the set of integers (resp. nonnegative integers).

A noetherian ring S is called a *standard graded algebra* if $S = \bigoplus_{n \geq 0} S_n$, where S_0 is a local ring with the maximal ideal $\mathfrak{n}$ and S is generated by S_1 as an algebra over S_0. We denote by $S_+ = \bigoplus_{n>0} S_n$ the ideal generated by all homogeneous elements of S of positive degree and $\mathfrak{M} = \mathfrak{n} \oplus S_+$ the homogeneous maximal ideal of S. (See [15] for information on noetherian graded rings.) For a graded S-module E denote by $[E]_n$ the nth graded piece of E. $E(p)$ denotes the same module E shifted by p, i.e. $[E(p)]_n = E_{n+p}$. In this paper we consider both local cohomology modules $H^i_{\mathfrak{M}}(E)$ and $H^i_{S_+}(E)$ with support in $\mathfrak{M}$ and (resp.) in S_+. Note that they are graded S-modules and $[H^i_{\mathfrak{M}}(E)]_n = [H^i_{S_+}(E)]_n = 0$ for all large n. Set

$$\underline{a}_i(S) = \sup\{n \in \mathbf{Z}; [H^i_{S_+}(S)]_n \neq 0\},$$

and

$$a_i(S) = \sup\{n \in \mathbf{Z}; [H^i_{\mathfrak{M}}(S)]_n \neq 0\}.$$

(Convention: if $H^i_{S_+}(S) = 0$ (resp. $H^i_{\mathfrak{M}}(S) = 0$) we set $\underline{a}_i(S) = -\infty$ (resp. $a_i(S) = -\infty$)). For convenience we denote $a_d(S)$ simply as $a(S)$, where $d = \dim S$. It is often called the *a-invariant* of S and was introduced by Goto and Watanabe [5]. The number

$$\operatorname{reg} S = \max\{i + \underline{a}_i(S); i \geq 0\},$$

is called *Castelnuovo–Mumford regularity* of S. We also use the following notation:

$$\operatorname{reg}_t S = \max\{i + \underline{a}_i(S); i \geq t\}.$$

Thus $\operatorname{reg} S = \operatorname{reg}_0 S$. The relationship between the reduction number $r_J(I)$ and the above-defined cohomological invariants of $G(I)$ is given in the following lemma:

Lemma 2.1 (Trung [23, Proposition 3.2]). *Let J be an minimal reduction of I and $s = l(I)$. Then*

$$\underline{a}_s(G(I)) + s \leq r_J(I) \leq \operatorname{reg}(G(I)).$$

Corollary 5.3 in this paper shows that we can often replace $\operatorname{reg} G(I)$ in the above inequality by $\operatorname{reg}_1 G(I)$. The following result can be derived from the proof Theorem 1.2 of [23] and is given explicitly in [19, Theorem 3.3]. It provides a condition when the reduction number is independent of the choice of a minimal reduction (cf. also [12]). Note that it is also an immediate consequence of the above lemma and Theorem 5.2 in this paper.

Lemma 2.2. *Assume that* $l(I) = \operatorname{grade} I = s$ *and* $\operatorname{grade} G(I)_+ \geq s - 1$. *Then*

$$r_J(I) = \underline{a}_s(G(I)) + s = \operatorname{reg}(G(I)),$$

for any minimal reduction J of I.

The a-invariant $a(G(I))$ plays an important role in studying the C-M and Gorenstein property of the Rees algebra $A[It]$ and the associated graded ring $G(I)$, see, e.g. [5, 9, 24]. We give here a result of Trung and Ikeda which will be used later.

Lemma 2.3 (Trung and Ikeda [24, Theorem 1.1]). *Let I be an ideal of A of positive height. Then $A[It]$ is a C-M ring if and only if $[H^i_{\mathfrak{M}}(G(I))]_n = 0$ for $i < d$, $n \neq 1$ and $a(G(I)) < 0$.*

In order to work with the invariants $\underline{a}_i(S)$ we need more information on the local cohomology modules with respect to S_+. We recall a result of Serre (see [8, Theorem III.5.2]):

Lemma 2.4. *Let S be a standard graded algebra and E a noetherian graded S-module. Then $[H^i_{S_+}(E)]_n$ is a noetherian S_0-module for all i and n.*

The following auxiliary result is of independent interest. In the ring case, it can be derived from [10, Lemma 2.3], by interpreting dim $S/\mathfrak{n}S$ as $l(S_+)$. In fact we do not need the notion of analytic spread, so we give here a proof along with the statement.

Lemma 2.5. *Let S be a standard graded algebra and E a noetherian graded S-module. Then*

$$\max\{i;\ H^i_{S_+}(E) \neq 0\} = (\text{Krull-})\dim_{S/\mathfrak{n}S} E/\mathfrak{n}E.$$

Proof. Let us denote by a and b the numbers on the left and on the right side of the above equation, respectively. In order to prove $a \le b$ we do induction on b. If $b = 0$ then there is n_0 such that $E_n = \mathfrak{n}E_n$ for $n > n_0$. Since E_n is a noetherian S_0-module, by Nakayama's lemma, $E_n = 0$ for $n > n_0$. By considering $S/\mathrm{Ann}_S(E)$ we can reduce to the case $S = S_0$ and conclude that $H^i_{S_+}(E) = 0$ for $i > 0$ (cf. [7, Proof of Proposition 6.4, p. 88]). Let $b > 0$. Take an element $x \in S_1$ such that $x \notin \mathfrak{p}$ for all homogeneous prime ideals $\mathfrak{p} \in Ass(E/\mathfrak{n}E) \cup Ass(E)$ with the property $S_+ \not\subseteq \mathfrak{p}$. Then $\dim (E/xE)/\mathfrak{n}(E/xE) = b - 1$ and $[0:_E x]_n = 0$ for $n \gg 0$. By the case $b = 0$ it follows that $H^i_{S_+}(0:_E x) = 0$ for $i > 0$. Hence, from the exact sequences

$$0 \to 0:_E x \to E \to E/(0:_E x) \to 0, \tag{1}$$

and

$$0 \to (E/(0:_E x))(-1) \xrightarrow{\cdot x} E \to E/xE \to 0, \tag{2}$$

we get by induction hypothesis the exact sequence:

$$0 \to [H^{b+1}_{S_+}(E)]_{n-1} \to [H^{b+1}_{S_+}(E)]_n,$$

for all $n \in \mathbf{Z}$. Since $[H^{b+1}_{S_+}(E)]_n = 0$ for $n \gg 0$, we must have $H^{b+1}_{S_+}(E) = 0$, i.e., $a \le b$.

Now, let us consider the exact sequence:

$$0 \to \mathfrak{n}E \to E \to E/\mathfrak{n}E \to 0.$$

Since $a \le b$, it gives the exact sequence:

$$H^b_{S_+}(E) \to H^b_{S_+}(E/\mathfrak{n}E) \to 0.$$

Note that $H^b_{S_+}(E/\mathfrak{n}E) \cong H^b_{S_+/\mathfrak{n}S_+}(E/\mathfrak{n}E) \neq 0$. Hence $H^b_{S_+}(E) \neq 0$, i.e. $a \ge b$, as required. □

3. Reduction number of $\mathfrak{M}$-primary ideals

In this section, let I be an $\mathfrak{m}$-primary ideal of A. If A is a C-M ring of multiplicity $e(A)$, then Sally [17, Theorem 2.2] showed that $r(\mathfrak{m}) \le d!e(A) - 1$. By using recent results of Shalev [20] we can give here an extension of this result of Sally to $\mathfrak{m}$-primary ideals in arbitrary local rings. If $d = 1$ and A is C-M, Eakin and Sathaye showed in the proof of Corollary 3 in [4] that $r(I) \le e(A) - 1$ (cf. [2, Propositions 2.2 and 2.3], where Achilles and Schenzel considered also the case A being a Buchsbaum ring). If $d > 1$ it is well-known that there is no bound on $r(I)$ which depends only on the entire ring A(take, for example, $I = (x^n, x^{n-1}y, y^n) \subset k[x, y]_{(x,y)})$. Hence, we need an additional, suitable invariant. Let

$$o(I) = \min\{n \in \mathbf{N};\ \mathfrak{m}^n \subseteq I\}.$$

$e(I)$ denotes the multiplicity of I and $e(A) = e(\mathfrak{m})$ (see [15, Section 14]). Let $v(I)$ denote the number of generators in a minimal basis of I. We have:

Proposition 3.1. *Let I be an* $\mathfrak{m}$-primary ideal of A.
(i) *If A is a C-M ring, then*

$$r(I) \le d!o(I)^{d-1}e(A) - 1 \quad \textit{and} \quad r(I) \le d!e(I)^{(d-1)/d}e(A)^{1/d} - 1.$$

(ii) *If A is a regular ring, then*

$$r(I) \le d \cdot o(I)^{d-1} - 1.$$

(iii) *For an arbitrary local ring A there is a constant B (depending only on A) such that*

$$r(I) \le B \cdot o(I)^{d-1}.$$

Moreover, for any given $\varepsilon > 0$ there is an integer $n(\varepsilon, A)$ such that for any ideal I with $o(I) \ge n(\varepsilon, A)$ we have

$$r(I) \le d(e(A) + \varepsilon)o(I)^{d-1} - 1.$$

Proof. For short, we set $t = o(I)$ and $e = e(A)$. The main theorem of Eakin and Sathaye in [4] states that for an $\mathfrak{m}$-primary ideal I in any local ring,

$$v(I^n) < \binom{n+d}{d}$$

implies $r(I) < n$. If A is a C-M ring, by [17, Theorem 1.2], $v(I^n) \le n^{d-1}t^{d-1}e + d - 1$. Hence the first inequality of (i) follows from the following inequality:

$$(d!t^{d-1}e)^{d-1}t^{d-1}e + d - 1 < \binom{d!t^{d-1}e + d}{d}.$$

Analogously, the second inequality of (i) follows from the following bound of Boratynski et al. [3, Theorem 5]:

$$v(I^n) \le [e(I^n)^{(d-1)/d}e(A)^{1/d}] + d - 1 = [n^{d-1}e(I)^{(d-1)/d}e(A)^{1/d}] + d - 1,$$

where $[x]$ denotes the integer part of x. Conclusion (ii) follows from the following result of Shalev [20, Corollary 4.3.1]): if A is a regular ring then

$$v(I^n) \le v(\mathfrak{m}^{nt}) = \binom{nt+d-1}{d-1}.$$

In order to prove (iii) we use [20, Theorem 4.3] which states that for any local ring

$$v(I) \le v(\mathfrak{m}^n) + D,$$

where n is any integer $\ge t$ and D depends only on the Hilbert function $H_A(n) = \ell(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. Let $r = r(\mathfrak{m})$. Then, for $n > \max\{r, D, v(\mathfrak{m}^r)dt^{d-1}\}$ one can check that

$$v(I^n) \le v(\mathfrak{m}^{tn}) + D \le v(\mathfrak{m}^r)v(\mathfrak{m}^{tn-r}) = v(\mathfrak{m}^r)\binom{tn-r+d-1}{d-1} + D < \binom{n+d}{d}.$$

Hence $r(I) \le Bt^{d-1}$, where $B = \max\{r, D, v(\mathfrak{m}^r)\}$.

Finally, let n_ε be a positive integer such that

$$H_A(n) \le (e+\varepsilon)\binom{n+d-1}{d-1},$$

for all $n \ge n_\varepsilon$. Let $n(\varepsilon, A) = \max\{(n_\varepsilon/d(e+1))^{1/d}, (D/d(e+1))^{1/(d-1)}\}$. Assume that $t \ge n(\varepsilon, A)$. Then, for $n = [d(e+\varepsilon)t^{d-1}]$ one can check that

$$v(I^n) \le v(\mathfrak{m}^{tn}) + D \le (e+\varepsilon)\binom{tn+d-1}{d-1} + D < \binom{n+d}{d}.$$

Therefore $r(I) \le n-1$, as required. □

4. Reduction number of equimultiple ideals

If I is an ideal and $\underline{x} = \{x_1, \ldots, x_t\}$ be a subset of elements of A we denote by $I/\underline{x}$ the ideal $I + (\underline{x})/(\underline{x})$ in the quotient ring $A/(\underline{x})$. Recall that an element $x \in I^s$ is said to be a *superficial element* of order s for I if there exists a positive integer c such that $(I^n : x) \cap I^c = I^{n-s}$ for all $n \gg 0$. An important property of superficial elements of order s for $\mathfrak{m}$ is that $e(A/xA) = s.e(A)$ if $\dim A \ge 2$ (see [26, Chapter 8]). $\underline{x} = \{x_1, \ldots, x_t\} \subset I$ is called a *superficial sequence* (of order 1) for I if x_i is a superficial element of order 1 for $I/(x_1, \ldots, x_{i-1})$ for all $1 \le i \le t$.

Lemma 4.1. *Assume that A is normally C-M along I, i.e. A/I^n is C-M for all $n \ge 1$. Let x be a parameter element of A modulo I and J a minimal reduction of I. Then J/x is a minimal reduction of I/x and $r(I) = r(I/x)$.*

Proof. Clearly, J/x is a reduction of I/x. Let $\bar{J} \subseteq J/x$ be a minimal reduction of I/x. One can write $\bar{J} = J'/x$ for some ideal $J' \subseteq J$. Let n be an nonnegative integer such that $(I/x)^{n+1} = (J'/x)(I/x)^n$. Then $I^{n+1} \subseteq J'I^n + (x)$. Since x is a regular element of A/I^n, it follows that $I^{n+1} \subseteq J'I^n + xI^{n+1} \subseteq J'I^n + (x^2) \subseteq \cdots$. By Krull's intesection theorem we get $I^{n+1} = J'I^n$, i.e. J' is a reduction of I. Hence $J' = J$ and J/x is a minimal reduction of I/x. In particular $r(I) \geq r(I/x)$. Further, let $r = r(J/x)$ and $\bar{J} = J'/x$ be a minimal reduction of I/x such that $(I/x)^{r+1} = \bar{J}(I/x)^r$, where $J' \subseteq I$. The above consideration shows that we can assume J' to be a minimal reduction of I, and $I^{r+1} = J'I^r$. Therefore $r(I) \leq r_{J'}(I) \leq r$ which gives $r(I) = r$. □

Using this lemma we can extend results of the previous section to the case of equimultiple ideals I such that A is normally C-M along I. For example, $r(I) \leq s!o(I/\underline{x})^{s-1}e(A) - 1$, where $s = \text{ht}\, I \geq 1$ and $\underline{x}$ is a s.o.p. of A modulo I such that $\underline{x}$ is a superficial sequence for $\mathfrak{m}$.

We will give in this section other bounds under the assumption that grade $G_+ \geq s - 1$. The following lemma is similar to Lemma 3.1 in [11].

Lemma 4.2. *Let S be a d-dimensional standard graded algebra with an artinian local ring S_0. Then*

$$a(S) + d \leq e(S_+) - 1.$$

Moreover, if S is a C-M ring, then

$$a(S) + d \leq e(S_+) - \ell(S_0).$$

Proof. We do induction on d. If $d = 0$ then $S = S_0 \oplus \cdots \oplus S_r$ with $S_i \neq 0$, $0 \leq i \leq r$. We have $a(S) = r \leq \ell(S) - \ell(S_0) = e(S_+) - \ell(S_0) \leq e(S_+) - 1$. Let $d > 0$. Consider $S' = S/H^0_{\mathfrak{M}}(S)$. Then $e(S'_+) = e(S_+)$. Take a nonzero divisor $x \in S'_1$ of S'. Set $S'' = S'/xS'$. The exact sequence

$$0 \longrightarrow S'(-1) \xrightarrow{\cdot x} S' \to S'' \longrightarrow 0,$$

gives $e(S''_+) = e(S'_+)$ and $a(S') + d \leq a(S'') + d - 1$. Hence, by induction hypothesis, we get: $a(S) + d = a(S') + d \leq e(S''_+) - 1 = e(S_+) - 1$. If S is a C-M ring, then S'' is again a C-M ring and $S''_0 = S_0$. Hence, again by induction, we obtain $a(S) + d \leq e(S) - \ell(S_0)$. □

Lemma 4.3. *Let I be an equimultiple ideal with $s = \text{ht}\, I > 0$. Assume that A/I^n is C-M for $n \gg 0$. Let $\underline{x} = \{x_1, \ldots, x_\delta\}$, $\delta = d - s$, be a s.o.p. of A modulo I. Then*

(i) $\underline{a}_s(G(I)) = \underline{a}_s(G(I/\underline{x}))$.

(ii) $\text{reg}_1\, G \leq \text{reg}_1\, G(I/\underline{x})$.

Proof. Let x be a parameter element of A modulo I. Consider the following exact sequences (note that $\deg(x^*) = 0$):

$$0 \to 0\colon x^* \to G \to G/(0\colon x^*) \to 0, \tag{3}$$

and

$$0 \to G/(0: x^*) \xrightarrow{x^*} G \longrightarrow G/x^*G \longrightarrow 0. \tag{4}$$

Since x is regular on A/I^n, $[0:x^*]_n = (I^{n+1}:x) \cap I^n/I^{n+1} = 0$ for $n \gg 0$. Hence, by Lemma 2.5, $H^i_{G_+}(0:x^*) = 0$ for $i > 0$. The exact sequence (3) gives $H^i_{G_+}(G) \cong H^i_{G_+}(G/(0:x^*))$ for $i > 0$. Also, by Lemma 2.5, we have $\max\{i; H^i_{G_+}(G) \neq 0\} = s$. Then, (4) induces the exact sequence

$$[H^s_{G_+}(G)]_n \xrightarrow{x^*} [H^s_{G_+}(G)]_n \longrightarrow [H^s_{G_+}(G/x^*G)]_n \longrightarrow 0. \tag{5}$$

From this it is immediate that $\underline{a}_s(G) \geq \underline{a}_s(G/x^*G)$. Let $n > \underline{a}_s(G/x^*G)$. Then the exact sequence (5) gives: $[H^s_{G_+}(G)]_n = x^*[H^s_{G_+}(G)]_n$. By Lemma 2.4 and Nakayama's lemma it follows that $[H^s_{G_+}(G)]_n = 0$. Hence $\underline{a}_s(G) \leq \underline{a}_s(G/x^*G)$, and so $\underline{a}_s(G) = \underline{a}_s(G/x^*G)$.

Now let us consider the following exact sequence of G-modules:

$$0 \to M \to G/x^*G \to G(I/x) \to 0, \tag{6}$$

where

$$M = \bigoplus_{n \geq 0} \frac{I^{n+1} + I^n \cap (x)}{I^{n+1} + xI^n} \cong \bigoplus_{n \geq 0} \frac{I^n \cap (x)}{xI^n + I^{n+1} \cap (x)}.$$

Since $I^n \cap (x) = x(I^n : x) = xI^n$ for $n \gg 0$, $[M]_n = 0$ for $n \gg 0$. Therefore, by Lemma 2.5, $H^i_{G_+}(M) = 0$ for $i > 0$. From (6) we then get $H^i_{G_+}(G/x^*G) \cong H^i_{G_+}(G(I/x))$ for $i > 0$ and

$$\underline{a}_s(G(I/x)) = \underline{a}_s(G/x^*G) = \underline{a}_s(G).$$

Moreover we have $\operatorname{ht}(I^n/x) \geq \operatorname{ht}(I^n + (x)) - 1$ [15, Theorem 13.6(ii)]. Since x is a regular element on A/I^n $(n \gg 0)$, $\operatorname{ht}(I^n + (x)) \geq \operatorname{ht}(I^n) + 1$. Therefore $s \geq l(I/x) \geq \operatorname{ht}(I/x) = \operatorname{ht}(I^n/x) \geq \operatorname{ht}(I^n) = s$. This shows that I/x is an equimultiple ideal of height s. Of course, $A/I^n + (x)$ is C-M for all $n \gg 0$. Hence, the equality (i) follows by induction.

Similarly, using instead (5) the exact sequence

$$[H^i_{G_+}(G)]_n \xrightarrow{x^*} [H^i_{G_+}(G)]_n \longrightarrow [H^i_{G_+}(G/x^*G)]_n, \quad i \geq 1,$$

we get $\underline{a}_i(G/x^*G) \geq \underline{a}_i(G)$ which yields (ii).

Theorem 4.4. *Let I be an equimultiple ideal in a C-M ring A with $s = \operatorname{ht} I \geq 1$. Assume that A/I^n is C-M for all $n \gg 0$ and grade $G(I)_+ \geq s - 1$.*

(i) *If A is an arbitrary local ring, then for any s.o.p. $\underline{x}$ of A modulo I we have*

$$r(I) \leq e(I/\underline{x}) - 1.$$

Moreover, if $G(I)$ is a C-M ring, then

$$r(I) \leq e((I/\underline{x}) - \ell(A/(I + (\underline{x})).$$

(ii) *If in addition $A = B_{\mathfrak{m}}$ and $I = J_{\mathfrak{m}}$, where B is a standard graded k-algebra, $\mathfrak{m} = B_+$ and J is a homogeneous ideal generated by elements of the same degree t, then*

$$r(I) \leq t^{s-1}e(A) - 1.$$

Proof. By Lemmas 2.2 and 4.3 we have $r(I) = \underline{a}_s(G(I/\underline{x})) + s$. Note that $I/\underline{x}$ is an $\mathfrak{m}$-primary ideal of $A/\underline{x}$ and $e(I/\underline{x}) = e(G(I/x)_+)$. Hence (i) follows from Lemma 4.2.

(ii) Similar to the local case, a homogeneous element $x \in B_s$ is called a superficial element of order s for a homogeneous ideal I if there exists a positive integer c such that $(I^n : x) \cap I^c = I^{n-s}$ for all $n \gg 0$. *Let* $\underline{x}$ be a homogeneous s.o.p. of B modulo J which is a superficial sequence for $\mathfrak{m}$. $\underline{x}$ consists of linear forms of B. Denote by $\underline{x}'$ the image of $\underline{x}$ in A. Then $\underline{x}'$ is a superficial sequence for $\mathfrak{m}A$, and $e(A/\underline{x}') = e(A)$. Again by Lemmas 2.2 and 4.3, $r(I) = a(G(I/\underline{x}')) + s$. Moreover $I/\underline{x}' \cong (J/\underline{x})_{\mathfrak{m}/\underline{x}}$, and $J/\underline{x}$ is generated by elements of degree t in $B/\underline{x}$. Therefore, in order to prove the second statement, it suffices to show that

$$a(G(I)) + d \leq t^{d-1}e(A) - 1,$$

where J is in addition an $\mathfrak{m}$-primary ideal.

For simplicity, we set in this proof $G = G(J)$. Note that $G(I) = G(J_{\mathfrak{m}}) = G_{(\mathfrak{M})}$, where $\mathfrak{M} = \mathfrak{m} + G_+$ and $M_{(\mathfrak{M})}$ denotes the *homogeneous localization* of a graded G-module M. Since

$$H^i_{G_{(\mathfrak{M})+}}(G_{(\mathfrak{M})}) \cong H^i_{G_{+(\mathfrak{M})}}(G_{(\mathfrak{M})}) \cong (H^i_{G_+}(G))_{(\mathfrak{M})},$$

$a(G(I)) = a(G_{(\mathfrak{M})}) = a(G)$. We define $e(B) = e(A)$. So, the second statement follows from the following inequality:

$$a(G) + d \leq t^{d-1}e(B) - 1,$$

where J is an $\mathfrak{m}$-primary ideal. To show the latter, we do induction on d. If $d = 1$ then the above inequality follows from Proposition 3.1 (i), since $a(G) + 1 = r(I)$. Let $d \geq 2$. We first prove the following claim:

Claim. *There exists a homogeneous element $y \in J$ of degree t such that y is superficial of order 1 for J as well as superficial of order t for $\mathfrak{m}$.*

Proof. Let $\mathfrak{p}_1 \dots, \mathfrak{p}_a$ be all homogeneous associated prime ideals of B. Assume that J is minimally generated by forms $f_1, \dots, f_b$ of degree t. Let $V = kf_1^* + \cdots + kf_b^* \subseteq J/J^2$. Since B is a C-M ring, ht $\mathfrak{p}_j = 0$ for $1 \leq j \leq a$. If $f_i^* \in \mathfrak{p}_j^*$, then $f_i + g \in \mathfrak{p}_j$ for some $g \in J^2$, which implies $f_i \in \mathfrak{p}_j$. Since ht $J > 0$, it follows that $V \not\subseteq \mathfrak{p}_j^*$ for $1 \leq j \leq a$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_c$ be all homogeneous associated prime ideals of G such that $G_+ \not\subseteq \mathfrak{q}_j$. Then $V \not\subseteq \mathfrak{q}_j$ for $1 \leq j \leq c$. Since k is infinite, there is an element $y^* \in V$ such that y^* is not in any of subspaces $V \cap \mathfrak{p}_i^*$ and $V \cap \mathfrak{q}_j$. Note that $G(\mathfrak{m}) \cong B$. Hence, if y is the homogeneous element of J which maps to y^*, then y satisfies the conclusion of the claim. $\square$

We now have $(0:y^*)_n = 0$ in G for $n \gg 0$. Hence, from the exact sequences

$$0 \to 0:y^* \to G \to G/(0:y^*) \to 0,$$

and

$$0 \longrightarrow G/(0:y^*)(-1) \xrightarrow{\cdot y^*} G \longrightarrow G/y^*G \longrightarrow 0,$$

we get that $a(G) + d \le a(G/y^*G) + d - 1$. On the other hand we have the exact sequence

$$0 \to M \to G/y^*G \to G(J/y) \to 0,$$

where

$$M = \bigoplus_{n \ge 0} \frac{J^{n+1} + J^n \cap (y)}{J^{n+1} + yJ^{n-1}} \cong \bigoplus_{n \ge 0} \frac{J^n \cap (y)}{yJ^{n-1} + J^{n+1} \cap (y)} \cong \bigoplus_{n \ge 0} \frac{J^n : y}{J^{n-1} + J^{n+1} : y}.$$

As y is a superficial element of order 1 for J, $J^n : y = J^{n-1}$ for $n \gg 0$ (see Remark 1 on p. 8 of [18]), and so $M_n = 0$ for $n \gg 0$. Since $\dim G/y^*G = \dim G(J/y) = d - 1 \ge 1$, from the above exact sequence we then get $a(G/y^*G) = a(G(J/y))$. Hence, by induction hypothesis we have

$$a(G) + d \le a(G(J/y)) + d - 1 \le t^{d-2} e(B/y) - 1.$$

Since y is a superficial element of order t for $\mathfrak{m}$, $e(B/y) = te(B)$. Thus $a(G) + d \le t^{d-1} e(B) - 1$, as required.

Remark and example (i) The bound $r(I) \le e(I) - \ell(A/I)$ was given by Schenzel [19, Theorem 4.4] for an $\mathfrak{m}$-primary ideal I provided that $G(I)$ is a C-M ring. This is no longer true for depth $G(I) = d - 1$. For example, take $I = (t^3, t^4) \subseteq A = k[[t^3, t^4, t^5]]$. Then $r(I) = 2 > e(I) - \ell(A/I) = 3 - 2$.

(ii) Let I be an $\mathfrak{m}$-primary ideal. Assume that depth $G(I) \ge d - 1$. It was stated in [23, Theorem 1.2] that $r(I) \le o(I)^{d-1} e(A) - 1$. Unfortunately its proof on p. 235 is based on a false claim: we can always choose an element $y \in I$ such that y is simultaneously a superficial element of order 1 for I as well as one (of some order) for $\mathfrak{m}$. This is true in the case (ii) of the above theorem. In the general case we have the following counter example: let $I = (t^{11}, s^{11}) \subset A = k[[t^4, t^5, t^{11}, s^4, s^5, s^{11}]]$. A is a C-M ring. Since t^{11}, s^{11} is a regular sequence, $G(I)$ is a C-M ring. Any superficial element u of order 1 for I must be an element in a minimal basis of a minimal reduction of I. Hence $u = at^{11} + bs^{11}$, where at least one of a and b does not belong to $\mathfrak{m}$. Without loss of generality assume that $a \notin \mathfrak{m}$. Then $u \notin \mathfrak{m}^2$. Hence the image of u^* in $G(\mathfrak{m})$ has the degree 1. Since $u^3 \in \mathfrak{m}^5$, $(u^*)^3 = 0$. This shows that u^* is not a parameter element of $G(\mathfrak{m})$. Therefore u cannot be a superficial element (of some order) for $\mathfrak{m}$. We do not know whether Trung's bound still holds for any local C-M ring. However, from the first statement of the theorem it follows that $r(I) \le o(I)^d e(A) - 1$ (cf. also Proposition 3.1).

(iii) Let $I = (x_1^n, x_1^{n-1}x_2, \ldots, x_1^{n-1}x_d, x_2^n, \ldots, x_d^n) \subseteq k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, where $d \ge 2$ and $n \ge 2$. Using [25, Corollary 2.7] one can check easily that $(x_2^n)^*, \ldots, (x_d^n)^*$ form a $G(I)$-sequence, i.e. depth $G(I) \ge d - 1$ (in fact $= d - 1$). $J = (x_1^n, \ldots, x_d^n)$ is a minimal reduction of I and $(x_1^{n-1}x_2)^{n-1} \cdots (x_1^{n-1}x_d)^{n-1} \notin (x_1^n, \ldots, x_d^n)I^{(d-1)(n-1)-1}$. Hence, by Lemma 2.2, $r(I) = r_J(I) = (n-1)^{d-1}$. This shows that for any dimension the bound of Theorem 4.4 (ii) is nearly sharp. It is attained by this example if $d = 2$. Note that $o(I) = d(n-1)$.

(iv) Let A be an equicharacteristic regular local ring and I an equimultiple ideal of ht $I = s \ge 1$. Assume that grade $G_+ = s$. Using an argument due to Huneke (see, for example, the introduction of [1]) we can show that $r(I) \le s - 1$. In fact, let $J = (x_1, \ldots, x_s)$ be a minimal reduction of I. By [21, Corollary 3.7], $I^s \subseteq J$. Since grade $G_+ = s$, $x_1^*, \ldots, x_s^*$ form a regular sequence in G. By [25, Corollary 2.7], $I^s = (x_1, \ldots, x_s) \cap I^s = (x_1, \ldots, x_s)I^{s-1}$. This observation suggests that there are much better bounds on $r(I)$ if grade $G_+ = s$.

As it was mentioned in the previous section, $r(I) \le e(A) - 1$ for any ideal of positive height in a one-dimensional C-M ring. Huckaba [13] asked whether this result is true for any regular ideal of analytic spread one in higher dimensions. Under some additional assumptions he proved there that $r(I) \le 1$ provided $e(A) \le 2$ (but A is not necessarily C-M). We give here another partial result.

Lemma 4.5. *Let I be an ideal of* ht $I = 1$ *in a C-M ring A. Assume that A/I^n is a C-M ring for some $n \ge e(A)$. Then $l(I) = 1$ and $r(I) \le e(A) - 1$.*

Proof. By a Rees's result (see, e.g., [17, Theorem 2.1 and Remark 1]),

$$v(I^n) \le e(A) < \binom{n+1}{1}.$$

Hence, by [4, the main theorem], there is an element $x \in I$ such that $I^n = xI^{n-1}$. Since x is a regular element it then follows that $v(I^{n-1}) = v(I^n)$. Again, by [4], $I^{n-1} = yI^{n-2}$ if $n - 1 \ge e(A)$. Repeating this process we get finally that $v(I^{e(A)}) \le e(A)$, and $r(I) \le e(A) - 1$. □

As a consequence we get an extension of [13, Proposition 3.5].

Corollary 4.6. *Let I be an ideal of* ht $I = 1$ *in a C-M ring A. Then A/I^n is a C-M ring for some $n \ge e(A)$ if and if A/I^n is a C-M ring for all $n \ge e(A) - 1$.*

Proof. We can assume that $\dim A \geq 2$. The if part if trivial while the only if part follows from Lemma 4.5 and the remark that A/I^i is C-M if and only if I^i is a C-M A-module. □

Corollary 4.7. *Let I be an ideal of height one in a C-M ring A. Then $G(I)$ is a C-M ring and I is equimultiple if and only if the following conditions are satisfied.*

(i) *A/I^n is C-M for all $1 \leq n \leq e(A)$.*
(ii) *For some (or all) minimal reduction (a) of I we have*

$$(a) \cap I^n = aI^{n-1} \quad \text{for} \quad 2 \leq n \leq e(A) - 1.$$

Proof. By Lemma 4.5, we can assume from the beginning that I is an equimultiple ideal. Let $\underline{x} = \{x_1, \ldots, x_{d-1}\}$ be a s.o.p. of A modulo I. Assume that G is C-M. By [6, Proposition 4.5], A/I^n is C-M for all $n \geq 1$, and

$$(a) \cap I^n \subseteq aI^{n-1} + (x_1^i, \ldots, x_{d-1}^i),$$

for all $n \geq 1$ and $i \geq 1$. Hence, by Krull's intersection theorem we get (ii) for all n. Converserly, assume that (i) and (ii) hold. Then, by Corollary 4.6, A is normally C-M along I. By Lemma 4.5, $(a) \cap I^n = aI^{n-1}$ for all $n \geq 1$ (note that $r_{(a)}(I)$ does not depend on the choice of (a)). Hence, by [6, Proposition 4.5, (iv) $\Rightarrow$ (i)], $G(I/\underline{x})$ is a C-M ring. Thus G is C-M by [6, Proposition 4.5, (ii) $\Rightarrow$ (i)]. □

5. Rees algebra

Marley [14, Theorem 2.1] proved that for an $\mathfrak{m}$-primary ideal I in a local C-M ring we have $\underline{a}_t(G) < \underline{a}_{t+1}(G)$, where $t = \operatorname{grade} G_+$ is assumed to be $\leq d - 1$. This implies, in particular, that in order to compute the upper bound reg G on $r_J(I)$ in Lemma 2.1 one needs only to consider $d - t$ local cohomology modules $H^i_{G_+}(G)$, $i > t$. We will give here an extension of this result, and then apply it to study the C-M property of the Rees algebra. We begin with a simple result.

Lemma 5.1. *Let x be an element in a minimal basis of a minimal reduction of I. Assume that x^* is a regular element on $G(I)$. Then $l(I/(x)) = l(I) - 1$.*

Proof. Set $l(I) = l$. Clearly that $l(I/(x)) \leq l - 1$. Assume that $l(I/(x)) < l - 1$. By Lemma 2.5, $H^{l-1}_{G_+}(G(I/(x)) = 0$. From the exact sequence:

$$0 \longrightarrow G(-1) \xrightarrow{x^*} G \longrightarrow G/x^*G \cong G(I/(x)) \to 0,$$

we then get an injection

$$0 \to [H^l_{G_+}(G)]_{n-1} \to [H^l_{G_+}(G)]_n,$$

for all $n \in \mathbf{Z}$. Since $[H^l_{G_+}(G)]_n = 0$ for $n \gg 0$, $H^l_{G_+}(G) = 0$ which is, by Lemma 2.5, a contradiction to $l = l(I)$. $\square$

Note that the above lemma is not true if there is no restriction on x^*. For example, the ideal $I = (u^2, uv) \subset k[u, v]_{(u,v)}$ has the analytic spread 2, but $I/(u^2)$ is a nilpotent ideal.

Theorem 5.2. *Let I be an ideal (not necessarily equimultiple) in an arbitrary local ring A. Assume that $t = \operatorname{grade} G(I)_+ < l(I)$ and $\operatorname{grade} I \geq t + 1$. Then $\underline{a}_t(G(I)) < \underline{a}_{t+1}(G(I))$.*

Proof. We do induction on t. Let $t = 0$. Consider two exact sequences:

$$0 \to R_+ \to R \to A \to 0,$$

and

$$0 \to R_+(1) \to R \to G \to 0.$$

Note that $H^i_{R_+}(G) \cong H^i_{G_+}(G)$. Since I contains a regular element, $H^0_{R_+}(R) = 0$. Further, $[H^i_{R_+}(A)]_n = 0$ for $n \neq 0$, so $[H^i_{R_+}(R_+)]_n \cong [H^i_{R_+}(R)]_n$ for $n \neq 0$. Hence, for $n \geq 0$ we get an exact sequence:

$$0 \to [H^0_{R_+}(G)]_n \to [H^1_{R_+}(R)]_{n+1} \to [H^1_{R_+}(R)]_n \to [H^1_{R_+}(G)]_n. \tag{7}$$

Set $b = \underline{a}_1(G)$. Then for $n \geq \max\{b + 1, 0\}$ we get from (7) an epimorphism:

$$[H^1_{R_+}(R)]_{n+1} \to [H^1_{R_+}(R)]_n \to 0.$$

Since $[H^1_{R_+}(R)]_{n+1} = 0$ for $n \gg 0$, we must have $[H^1_{R_+}(R)]_n = 0$. If $b \leq 0$, from (7) it then follows that $[H^0_{R_+}(G)]_n = 0$ for $n \geq 0$. But $H^0_{R_+}(G)$ is an ideal of G, we also have $[H^0_{R_+}(G)]_n = 0$ for $n < 0$. Hence $H^0_{R_+}(G) = 0$ which contradicts to the assumption $t = 0$. Thus $b \geq 1$. Considering again the exact sequence (7) we see that $[H^0_{R_+}(G)]_n = 0$ for $n \geq b$. This means $\underline{a}_0(G) \leq b - 1 < \underline{a}_1(G)$.

Now let $t > 0$. Let x be an element in a minimal basis of a minimal reduction of I such that x^* is a regular element in G (such a choice is always possible). Then x is a regular element in A and, by Lemma 5.1, $l(I/(x)) = l - 1$. Moreover $\operatorname{grade} G(I/(x))_+ = t - 1$ and $\operatorname{grade} I/(x) \geq t$. Set $\bar{G} = G(I/(x)) \cong G/x^*G$. The exact sequence

$$0 \to G(-1) \xrightarrow{x^*} G \to \bar{G} \to 0$$

induces the following exact sequence:

$$0 \to [H^{t-1}_{G_+}(\bar{G})]_n \to [H^t_{G_+}(G)]_{n-1} \to [H^t_{G_+}(G)]_n \to [H^t_{G_+}(\bar{G})]_n$$
$$\to [H^{t+1}_{G_+}(G)]_{n-1}. \tag{8}$$

By the induction hypothesis, $\underline{a}_t(\bar{G}) > \underline{a}_{t-1}(\bar{G})$. In particular, $\underline{a}_t(\bar{G})$ is an integer (since grade $\bar{G}_+ = t - 1$). Let $n \geq \underline{a}_t(\bar{G})$. From (8) we then get an injection:

$$0 \to [H^t_{G_+}(G)]_{n-1} \to [H^t_{G_+}(G)]_n.$$

Hence $[H^t_{G_+}(G)]_{n-1} = 0$ for all $n \geq \underline{a}_t(\bar{G})$, i.e. $\underline{a}_t(G) < \underline{a}_t(\bar{G}) - 1$. But then, setting $n = \underline{a}_t(\bar{G})$ in (8) it follows that $[H^{t+1}_{G_+}(G)]_{\underline{a}_t(\bar{G})-1} \neq 0$, i.e., $\underline{a}_{t+1}(G) \geq \underline{a}_t(\bar{G}) - 1$. Thus $\underline{a}_{t+1}(G) > \underline{a}_t(G)$, as required. □

Corollary 5.3. *Assume that I contains a regular element. Then* $\operatorname{reg} G(I) = \operatorname{reg}_1 G(I)$.

Now we can prove a converse of [9, Proposition 45.4].

Theorem 5.4. *Let I be an equimultiple ideal of a local ring A and $s = \operatorname{ht} I > 0$. Then $A[It]$ is a C-M ring if and only if the following conditions are satisfied:*
(i) $r(I) \leq \operatorname{ht} I - 1$.
(ii) $H^i_{\mathfrak{M}}(G)]_n = 0$ *for* $n \neq -1$ *and* $i < d$.

Proof. Assume that $A[It]$ is C-M. Then (ii) follows by Lemma 2.3. (i) is proved in [9, Proposition 45.4]. We give here another proof of (i). Let $\underline{x}$ be a s.o.p. of A modulo I. By [24, Theorem 5.3], the Rees algebra $A/(\underline{x})[(I/\underline{x})t]$ is also C-M and A is normally C-M along I. Hence, from Lemmas 4.1, 2.1, and 2.3 we get

$$r(I) = r(I/\underline{x}) = \operatorname{reg} G(I/\underline{x}) = \max\{i + a_i(G(I/\underline{x}))\} \leq s - 1.$$

For the converse, by Lemma 2.3 it suffices to show that $a(G) < 0$. Let us see how $a(G)$ changes by cutting by x_1^*. Note that $\deg(x_i^*) = 0$. From the exact sequence

$$0 \to G/(0:x_1^*) \xrightarrow{x_1^*} G \to G/x_1^*G \to 0,$$

we get an exact sequence:

$$0 = [H^{d-1}_{\mathfrak{M}}(G/x_1^*G)]_n \to [H^d_{\mathfrak{M}}(G)]_n \xrightarrow{x_1^*} [H^d_{\mathfrak{M}}(G)]_n \to 0,$$

for all $n > a(G/x_1^*G)$. In particular, x_1^* is a nonzero divisor of $[H^d_{\mathfrak{M}}(G)]_n$. Take $u \in [H^d_{\mathfrak{M}}(G)]_n$. It is known that $(x_1^*)^p u = 0$ for some $p \gg 0$. Hence $u = 0$ and $[H^d_{\mathfrak{M}}(G)]_n = 0$. This shows that $a(G) \leq a(G/x_1^*G)$. Repeating this process we get $a(G) \leq a(\bar{G})$, where $\bar{G} = G/(\underline{x}^*)G$. Let J be a minimal reduction of I such that $r_J(I) = r(I)$. Then J^* is a minimal reduction of G_+ and $r_{J^*}(G_+) = r_J(I)$. Note that, by [9, Proposition 10.20], G_+ is a $\mathfrak{M}\bar{G}$-primary ideal with $\operatorname{ht} \bar{G}_+ = \operatorname{ht} G_+ = \operatorname{ht} I$. In this case it follows from [9, Corollary 10.15] that $J^*/(\underline{x}^*)$ is also a minimal reduction of $\bar{G}_+$ (we do not need the assumption of Lemma 4.1). Hence $r(\bar{G}_+) \leq r(G_+) = r(I)$. By Lemma 2.1 and (i) we then get

$$a(G) \leq a(\bar{G}) = \underline{a}_s(\bar{G}) \leq r(\bar{G}_+) - s \leq r(I) - s < 0. \quad \square$$

This theorem extends the following results of Goto–Shimoda (for $\mathfrak{m}$-primary case) and of Grothe–Herrmann–Orbanz [6, Theorem 4.8]. It is also an immediate consequence of Lemmas 2.3 and 4.1. Note that, by a well known result of Huneke, if A is a C-M ring then the C-M property of $A[It]$ forces $G(I)$ to be C-M.

Corollary 5.5. *Let A be a C-M ring and I an equimultiple ideal of A with* $\operatorname{ht} I \geq 1$. *Then $A[It]$ is C-M if and only if* $G(I)$ is C-M and $r(I) \leq \operatorname{ht} I - 1$.

Assume that I is an equimultiple ideal in a local C-M ring A. As we have mentioned before, the reduction number $r_J(I)$ is independent from the choice of a minimal reduction J provided that $G(I)$ is C-M. It turns out that this is also true if $A[It]$ is C-M, but A is not necessarily C-M (if A is a C-M ring, the C-M property of $A[It]$ would imply the C-M property of $G(I)$ and there is nothing new).

Proposition 5.6. *Let I be an equimultiple ideal of A. Assume that $A[It]$ is C-M. Then $r(I) = \operatorname{reg}(G(I))$. In particular, the reduction number $r(I)$ is independent from the choice of minimal reduction.*

Proof. We can assume that $s = \operatorname{ht} I > 0$. Let $J = (x_1, \dots, x_s) \subseteq I$ be a minimal reduction of I. Then $J^* = (x_1^*, \dots, x_s^*)$ is a minimal reduction of G_+ and $r_J(I) = r_{J^*}(G_+)$. First, let us consider the case of $\mathfrak{m}$-primary ideals, i.e. $s = d$. By Lemma 2.3, $[H^i_{\mathfrak{M}}(G)]_n = 0$ for $i < d$ and $n \neq -1$. Hence, by [22, Corollary 3.12], G_+ is a standard ideal of G. By [22, Proposition 3.1], it implies that $x_1^*, \dots, x_d^*$ is a d-sequence. In this case, by [23, Corollary 3.3 and Lemma 3.4], we have $r_{J^*}(G_+) = \operatorname{reg} G$. This means that $r_J(I)$ does not depend on the choice of J and $r(I) = \operatorname{reg} G$.

Now, let $s < d$. Since It contains a homogeneous parameter element of $A[It]$ (see [6, Proposition 2.6]), I satisfies the assumption of Corollary 5.3. Therefore $\operatorname{reg} G = \operatorname{reg}_1(G)$. Further, by [24, Theorem 5.3], A/I^n and $A/(\underline{x})[(I/\underline{x})t]$ are C-M rings for all $n \geq 1$. By Lemmas 4.3 (ii) and 2.1 we then get

$$r_J(I) \leq \operatorname{reg} G = \operatorname{reg}_1 G \leq \operatorname{reg}_1 G(I/\underline{x}) \leq \operatorname{reg}(G/\underline{x})) = r_{J/\underline{x}}(I/\underline{x}).$$

The last equality follows from the case of $\mathfrak{m}$-primary ideals and the fact that $J/\underline{x}$ is also a minimal reduction of $I/\underline{x}$ (Lemma 4.1). It is obvious that $r_{J/\underline{x}}(I/\underline{x}) \leq r_J(I)$. Hence $r(I) = r_J(I) = \operatorname{reg} G$, as required. □

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